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GL-TR-89-0309

A Mathematical Appreciation of Antonio Marussi's Contributions
to Geodesy

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November 20, 1989

Scientific Report No. 2

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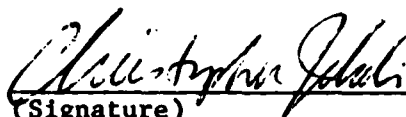HANSCOM AIR FORCE BASE, MASSACHUSSETTS 01731-5000
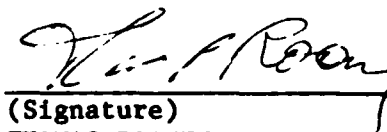
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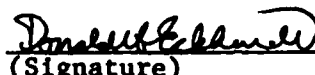(Signature)
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Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	20 November 1989	Scientific Report No. 2

4. TITLE AND SUBTITLE	5. FUNDING NUMBERS
A Mathematical Appreciation of Antonio Marussi's Contribution to Geodesy	61102F 2309G1BZ Contract F19628-89-K-0044

6. AUTHOR(S)

J. D. Zund

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	8. PERFORMING ORGANIZATION REPORT NUMBER
New Mexico State University Department of Mathematical Sciences Las Cruces, NM 88003-0001	

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSORING/MONITORING AGENCY REPORT NUMBER
Geophysics Laboratory Hanscom AFB Massachusetts 01731-5000 Contract Manager: Christopher Jekeli/LWG	GL-TR-89-0309

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT	12b. DISTRIBUTION CODE
Approved for public release; distribution unlimited	

13. ABSTRACT (Maximum 200 words)

This report contains an appreciation of the contribution of A. Marussi to geodesy. It includes a discussion of his notion of intrinsic geodesy, his mathematical methods, and an introduction to the Marussi Hypothesis.

14. SUBJECT TERMS		15. NUMBER OF PAGES
A. Marussi, intrinsic geodesy, Marussi Hypothesis		26
		16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	SAR

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Preface

The following manuscript is based on a presentation made at the Marussi Commemorazione, or Geodetic Day, at the Accademia dei Lincei in Rome on October 9, 1989. It is the custom of the Accademia to honor its most distinguished members approximately five years after their death with such a commemorative session. The session was organized by Fernando Sansò and Michele Caputo, and the President of the Accademia Edoardo Amaldi. It consisted of nine invited speakers, and the program was as follows:

J.D. Zund (United States):
  "The Scientific Work of A. Marussi;"

F. Bocchio (Italy):
  "Geometry, Topology, and the Gravity Field;"

H. Moritz (Austria):
  "The Equilibrium Figure of the Earth;"

M. Caputo (Italy):
  "Rheology and Geodynamics;"

E. Grafarend (Federal Republic of Germany):
  "Relativistic Geodesy;"

F. Sansò (Italy):
  "The Boundary Value Problems of Geodesy;"

T. Krarup (Denmark):
  "Approximating the Functionals of the Gravity Field;"

R. Rummel (The Netherlands):
  "The Gravity Field Measured from Space;"

E. Livieratos (Greece):
  "Geodesy and Geodynamics."

Due to the number of speakers, each presentation was limited to thirty-forty minutes, and since my talk was an abbreviated version of the prepared manuscript, I have reverted in this report to my original title. It is conceivable, but by no means certain at this time, that the Accademia dei Lincei may eventually publish the proceedings of the Marussi Commemorazione.

Although much of my presentation was intended to be historical, and an appreciation of Marussi's contributions to mathematical geodesy, Sections 2 and 5 contain an introduction to his notion of intrinsic geodesy and the Marussi Hypothesis. The latter is the principal topic of my research contract with the Geophysical Laboratory, and this presentation was my first discussion of this material. At the Fall Meeting of the American Geophysical Society (San Francisco, December 6, 1989) I presented an abstract, "The Marussi Hypothesis in Differential Geodesy," which will also deal with the material in Sections 3 and 5 of this report.

1. Introduction.

A great master of modern geodesy passed away when Antonio Marussi died in Trieste on April 24, 1984 at the age of seventy-six. Marussi not only enriched geodesy by his conception of intrinsic geodesy, but he also impressed the spirit of his genius and youthful vigor on the entire subject of theoretical geodesy. *His vision and enthusiasm fired the imagination of* Martin Hotine (1898-1968), and together they formulated the Marussi-Hotine approach to geodesy. Although today this is only one aspect of research in geodesy, it is impossible to think of mathematical geodesy without immediately bringing to mind these two great men and the theory they labored to perfect. In Marussi's case this is even more remarkable, when we recognize that his contributions to geodesy are but *one* aspect of his creativity which freely ranged over the broader field of geophysics. As Sir Alan Cook wrote in his tribute in [1], truly

"Antonio Marussi bestrode the world like a colossus."

In this lecture, we present a mathematical appreciation of Marussi's contributions to geodesy. First, we consider his mathematical and geodetic background, and then his conception of intrinsic geodesy which we regard as

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his most important and lasting contribution to geodesy. We then discuss the mathematical methods he employed in the formulation of his ideas, and how his work stands today from the standpoint of differential geodesy.

2. The Mathematical and Geodetic Background of Marussi.

Marussi was born in Trieste and all of his life he was deeply attached to this city and its rich cultural, ethnic, and historical heritage. As a student there he attended the Scuola Scientifica Galileo Galilei where he excelled in his study of mathematics. His university studies began at the Istituto Politecnico di Milano (1926) where he initially intended to become an engineer. However, during his first year of study there his interests were shifted to mathematics by Oscar Chisini (1889-1967) whose lectures on analytic and projective geometry caused him to 'fall in love with geometry.' These mathematical interests were further stimulated by the lectures of Bruno Finzi (1899-1974) on calculus and algebra. Marussi then transferred to the Università di Bologna, where he pursued his mathematical studies from 1927 to 1931. The Bologna school was led by a Triestino, Salvadore Pincherle (1853-1936), who was both a dynamic researcher and teacher. Marussi took many courses from Pincherle and was deeply influenced by his teaching as well as by his aesthetic and cultural interests. Among the many mathematical luminaries in Bologna were Leonida Tonelli (1885-1946), Giuseppe Vitali (1875-1932), Enea Bortolotti (1896-1942), and Pietro Burgatti (1868-1938). There is no doubt that Marussi received a first-class mathematical education in Bologna. He completed his studies with a Tesi di laurea in pure mathematics: *I sistemi d'equazioni alle derivate parziali composti di tante equazioni quante funzioni incognite* (July 1931) written under Vitali's direction. The thesis dealt with a difficult question in the Riquier-Janet theory of systems of partial differential equations and the attempt to analyze the system of

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characteristics defined by them. Essentially it consisted of a sequence of observations on the analytic difficulties, and suggestions for remedying them. It was not published, and even today many questions in the Riquier-Janet theory remain unanswered.

Although the thesis was a creditable piece of research, with Vitali's premature death, it was probably wise on Marussi's part that he did not choose to continue a mathematical career in this area. It was not one which was ripe for a breakthrough, or one in which someone could readily establish a reputation. Moreover, it had only a tenuous connection with geometry, and while one might well imagine Marussi becoming a professional geometer it was probably not in his temperament to embark on a career devoted to proving delicate and difficult existence and uniqueness theorems. In retrospect, he was probably as grateful as we are that he did not pursue a career in pure mathematics.

Upon graduation, Marussi had two passions: mathematics (i.e., geometry) and mountains (i.e., mountaineering). One evening, while crossing the Piazza dell'Unità d'Italia in Trieste, he met a friend from his alpine group who told him of an advertisement for a position as a geographical engineer at the Istituto Geografico Militare (I.G.M.) in Firenze. This seemed ideal to him since it would combine his fascination for maps, viz geometry in the guise of cartography, with mountains, since a new survey of the Alps was being planned. Thus, he began his long and fruitful association with the I.G.M., which, apart from brief interruptions for military service and work as an actuary at the Assicurazioni Generali in Trieste, was to lead him to surveys in Ethiopia in 1936 and during the later war years in Albania and Greece. He remained at the I.G.M. until 1952 when he accepted a position at the Università di Trieste as Professor of Geodesy and Geophysics and founder (and for many years the director) of the Istituto di Geodesia e Geofisica at this university.

Marussi's association with the I.G.M. gave him valuable practical experience which gradually led him from surveying and applied cartography to geodesy. As he told Ian Reilly (in a taped interview in 1982)

> "I come from practical surveying, and this is of great importance to my career; and I can see both the problems of the practical surveyor and the cartographer, and the theoretical man that wishes to have the ideas absolutely clear. I am very grateful for what I have learned in practical surveying, and this gives me an idea of what practical work means."

When he came to the I.G.M. he had no formal training in surveying, cartography, or geodesy. He embarked on an extensive program of self-study which included the classical texts of Helmert and Jordan-Eggert. As he later was to tell Reilly (loc. cit. supra) in geodesy he was 'a self-made man totally.' As his studies progressed he came across the papers of Corradino Mineo (1875-1960) which 'positively influenced' his line of thought. Marussi's ideas on geodesy required a fifteen year period of gestation and did not appear in print until he was almost forty years old. Curiously enough, Mineo became a vociferous critic of intrinsic geodesy in his later years, despite Marussi's acknowledgement of his debt to him.

3. The Notion of Intrinsic Geodesy.

In his study of the geodetic literature, Marussi found many concepts and relations, such as that between the geoid and the ellipsoid of reference, to be unclear and unsatisfying. He also felt the usual two-dimensional treatment of geodesy to be both artificial and unnatural. Guided by his practical experience and his pure mathematical training, he set out to remedy these deficiencies and he sought a unified point of view which would provide a rational foundation for geodesy. As the American theoretical physicist J.W.

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Gibbs (1881) wrote

> "One of the principal objects of theoretical research in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity."

From his love of geometry, Marussi naturally found this point of view. He discovered that the traditional definition of H. Bruns (1878) and F. Helmert (1880) that geodesy was

> "... the science of the measurement and mapping of the Earth's surface,"

could be reformulated by regarding

> "Geodesy is the science which is devoted to the study of the Earth's gravity field."

Thus, when the gravity field of the Earth is described in potential-theoretic terms, geodesy is merged with the differential geometry of the equipotential surfaces of the Earth's gravity field. However, it is more than merely differential geometry since it involves reconciling purely mathematical quantities with the process of doing geodetic measurements on and between the equipotential surfaces. In effect such a procedure seeks to enhance the physical description of the gravity field by attempting to recast it in geometric terms.

In formulating his approach to geodesy Marussi made a number of basic assumptions which we state as follows:

(i) the geometry of space is Euclidean and three-dimensional;

(ii) the equipotential surfaces of the Earth's gravity field are locally isometrically imbedded in a three-dimensional Euclidean space;

(iii) the choice of reference systems should be natural and not contrived;

(iv)   the reference systems should involve no additional hypotheses or otherwise impose any loss of generality in our description of the gravity field;

(v)   the reference systems employed in the analysis must be susceptible of an immediate physical interpretation;

(vi)   the components of all vectors/tensors occurring in the theory should be readily measurable;

(vii)   the domain of the reference systems must be sufficiently large to be useful, viz to allow one to make measurements and give a description of the gravity field in a required vicinity;

(viii) the domains of various reference systems should be continuable, or extendable, in the sense that they are compatible and one can readily pass between neighboring systems of reference.

We have stated these requirements in a form which is slightly more general than that given by Marussi. Originally he assumed that the reference systems were coordinate systems, but later he extended them to include systems of linearly independent vectors, i.e. the *leg systems* in the terminology of E.W. Grafarend (1986). The requirements (i) and (ii) were implicit in his mathematical formulation of his theory, and (iii) is the origin of his term 'intrinsic.'   Requirements (iv)-(vi) constituted his basic physical assumptions, while (vii) and (viii) were implicit in his considerations. In practice, not all of these requirements are rigorously satisfiable, but in order of importance -- after (i) and (ii) -- the requirements (iii) and (iv) were crucial and it was assumed that such systems *exist* and that (iii) and (iv) would expedite the fulfillment of (v) and (vi). Requirements (vii) and (viii) are highly desirable, however, they clearly depend on the physical situation under consideration. Note that (vii) and (viii) are reminiscent of the conditions occurring for coordinate neighborhoods of a differentiable

manifold except that no differentiability, i.e. smoothness, conditions were imposed.

Needless to say, Marussi only partially succeeded in demonstrating that in practice all these requirements could be rigorously satisfied. Indeed, rather than rigid requirements, he probably regarded them as being guidelines for choosing reference systems. No one has done any better in this respect, and few have done as well. Almost thirty years later N. Grossman (1979) observed that

> "The predominant view is that space near the Earth is a manifold. The unpleasant fact of life is that no one has ever described a method for coordinatizing that supposed manifold in a way consonant with physical reality."

Thus, the above requirements are highly non-trivial and intended to yield not only an aesthetically pleasing mathematical theory, but also a physical theory in which measurements can be made and interpreted. They are remarkable for their bold synthesis of geometric and physical requirements. Individually each is obvious, but taken together it is less than obvious that, possibly apart from Newtonian dynamics, they are ever rigorously satisfiable in practice. In effect, they ask for a physical theory which deals *only* with physically measurable quantities! However, if it is granted that Newtonian dynamics comes close to satisfying them, then a merger of it with the Newtonian theory of gravitation (when reformulated in potential-theoretical terms) and recast in a geometric setting, offers us a reasonable hope for success. This was precisely what Marussi offered us in his formulation of geodesy. In effect, it was a microcosm of what a physical theory might be if we could only achieve it. In this sense, if geometry is regarded as the most perfect of the mathematical sciences, then geodesy satisfying Marussi's requirements would play an analogous role in the physical sciences.

For purposes of discussion we will refer to Marussi's original theory as *intrinsic geodesy*, and reserve the term *differential geodesy* for a more general theory. Thus, intrinsic geodesy is a coordinate-based theory in which *intrinsic coordinates* play the primary role. On the other hand, differential geodesy deals with leg systems, or exterior differential forms, and includes intrinsic geodesy when the leg vectors coincide with the coordinate axes. In differential geodesy coordinates need not be of primary interest, and the resolutions of vectors/tensors in an appropriate leg system are the significant quantities. A simple example of such a situation is given by considering the motion of a particle along a space curve C. One takes the leg system to be the Frenet leg consisting of the tangent t, the principal normal n and the binormal b of C. Then relative to this 3-leg, the *velocity* v has the leg components:

$$v_t = \upsilon \ , \quad v_n = 0 \ , \quad v_b = 0 \ ;$$

and analogously for the *acceleration* $a = \dot{v}$ and *jerk* $j = \dot{a}$, we have

$$a_t = \dot{\upsilon} \ , \quad a_n = \kappa \upsilon^2 \ , \quad a_b = 0 \ ;$$

$$j_t = \ddot{\upsilon} - \kappa^2 \upsilon^3 , \quad j_n = 3\upsilon\dot{\upsilon}\kappa + \upsilon^2\dot{\kappa} \ , \quad j_b = \kappa\tau\upsilon^3 \ ;$$

here υ is the speed, the dot denotes differentiation with respect to time, and κ, τ are the curvature, torsion of C.

Marussi first presented his intrinsic geodesy in [2] which was probably the basis of his presentation to the General Assembly of the International Association of Geodesy in Oslo in 1948. It was given in Section V – The Geoid, under the Presidency of J. de Graaf-Hunter, with G. Bomford serving as Secretary. In retrospect, it was an all star program which also included presentations by Bomford and de Graaf-Hunter (both of Great Britain), W. Heiskanen (Finland), F.A. Vening Meinesz (The Netherlands) and A. Prey, K.

Mader (Austria). Later Bomford gave the following report[1] of Marussi's presentation:

> "Professor A. Marussi gave a summary of his *Fondements de géométrie différentielle absolue du champ potentiel terrestre*. This paper gives an account of the possible application of the methods of vector analysis to the study of the earth's gravitational field and of the simplications which may be derived from its use. It concludes that the most useful programme of work would be to make such observations of gravity, the deviations of the vertical, and their gradients, as can best determine the second derivatives of the potential at intervals of (say) 10 minutes of latitude and longitude over a large area."

While this is probably a fair assessment of its content, it is clear that Bomford did not recognize, or appreciate, the revolutionary aspects of Marussi's appraoch. Indeed it is possibly noteworthy, that Bomford never included *any* of Marussi's publications in the *extensive bibliography of* various editions of his celebrated book (1952-1980). Nevertheless, and despite his comment to Reilly (loc. cit. supra) that

> "... nobody followed it, but it was my fault since my symbols (i.e., his vectorial and tensorial methods) were not commonly understood,"

it was a significant occasion. It was at this meeting that he met Martin Hotine. In an obituary notice on Hotine, [3], Marussi wrote

> "... Martin said he understood only very little about it, but that it broke with crystallized tradition and that it must therefore be important."

---

[1]Bulletin Géodésique, April 1949, 75-77.

The Oslo lecture was followed by a veritable flurry of publications: a conference in Trieste [4]; a more substantial version [5] and a précis of the theory [6] both appearing in *Bulletin Géodésique*. In *toto*, Marussi published eight papers related to intrinsic geodesy in 1950, and this was followed by about a half-dozen papers during each of the years 1951, 1952, and 1953. The most significant of these are re-printed in [1], and these include his monumental [7] which he personally regarded as his definitive exposition of intrinsic geodesy. Of particular interest is the report, [8], which is based on lectures given in the United States in 1951-52. These are noteworthy since they present not only an overview of his work during the peak of his geodetic activity, but also the most detailed treatment of his mathematical methods which were essentially assumed *known* in his more formal publications.

Ultimately Marussi wrote about fifty papers (slightly under half of his publications) on topics related to intrinsic geodesy. In addition to laying out the fundamentals of his theory, he pioneered new developments in the theory of conformal mapping of the gravity field and between surfaces, the propagation of light in continuous isotropic refracting media, and questions dealing with satellite geodesy. A complete analysis of these contributions is beyond the scope of this lecture, and we must content ourselves to referring the reader to [1] where most of these studies appear.

4. Marussi's Mathematical Methods.

We now come to the methods of how Marussi chose to mathematically formulate his intrinsic geodesy. Such methods necessarily involve a formalism capable of handling the classical differential geometry of curves and surfaces in a three-dimensional Euclidean space. The obvious choice is to employ some form of vector/tensor calculus, and actually Marussi used both vectors and tensors. However, the precise variants of these methods which he employed

were unusual and did not tend to make his theory transparent even to those goedesists who had some prior knowledge of vectors and tensors. In order to understood his methods and motivation in choosing them, it is necessary to make a brief digression into the historical development of the calculus of vectors and tensors.

The theory of vectors essentially arose as a byproduct of the work of two men: W.R. Hamilton (1805-1863) and H. Grassmann (1809-1877). Both of them wrote weighty and almost unreadable treatises which included vectors as special cases of *quaternions* and *extensive magnitudes* respectively. Surprisingly enough, they both attracted small, but vocal, groups of enthusiastic partisans who saw in these theories an ideal formalism for handling problems arising in the physical sciences. While both theories were truly remarkable for their mathematical richness (Hamilton's theory involved a general theory of linear operators, whereas that of Grassmann contained his exterior algebra and tensors) neither won widespread support from physicists.

In the early 1880's, J.W. Gibbs (1839-1903) and O. Heaviside (1850-1925) independently succeeded in extracting from these theories, a simple and easily learned theory of vectors which constitutes what we now call *vector calculus*. Moreover, they immediately demonstrated the utility of their theory in Newtonian dynamics and Maxwellian electrodynamics. Physicists were gradually attracted to this new calculus; however, the supporters of Hamilton and Grassmann were appalled at what they regarded as a cannibalization of their work. A thirty year war was fought over which theory was superior, and it is one of the most fascinating chapters in the history of mathematics. In 1859 Hamilton wrote to his disciple P.G. Tait (1831-1901):

> "Could anything be simpler and more satisfactory? Don't you *feel*, as well as think, that we are on the *right track*, and shall be *thanked* hereafter? Never mind when."

Later Tait denounced the Gibbs-Heaviside theory as a 'hermaphrodite monster.' On the other side, in 1892, Heaviside countered that

> "Quaternions furnishes a uniquely simple and natural way of treating *quaternions*. Observe the emphasis."

By the turn of the century this *Kampf ums Dasein* came to an end with the death of the most combative quaternionists, and the Gibbs-Heaviside theory won the day. In retrospect, both camps were partially correct: the Gibbs-Heaviside theory is mathematically unsatisfying despite its utility. It is Cartesian, three-dimensional, and literally nothing more than a convenient set of computational rules. On the other hand despite its mathematical richness, quaternions are cumbersome and, in contrast to vectors, are essentially 'unphysical' quantities.

After the years of controversy the last thing most people wanted was a new and more mathematically sound vectorial formalism. However, this is precisely what was proposed by two ingenious Italians: the mathematician C. Burali-Forti (1861-1931) and the physicist R. Marcolongo (1862-1943). In 1912-13 they produced a two volume *Analyse vectorielle générale*, and ultimately they intended a multi-volume encyclopedic treatise *Analisi vettoriale generale e applicazioni* 1929-1930 which was not completed. The latter was written in collaboration with one of Marussi's Bologna professors, Pietro Burgatti, and Tommaso Boggio (1877-1963). The Burali-Forti and Marcolongo theory, which is more commonly known as the *homographic calculus*, was a mathematically elegant theory which reformulated the Gibbs-Heaviside vector calculus along quaternionic lines and *without* the *ab initio* intrusion of Cartesian notions. It was controversial -- partially due to the sharp tongue of Burali-Forti -- and even in the country of its origin it attracted few practitioners. However, Marussi learned it from Burgatti, and attracted by its elegance chose it as the *ideal method* for presenting his ideas on

intrinsic geodesy. We will return to a discussion and vindication of this choice in §4. However, to many geodesists of his time, including Mineo, this choice was not met with enthusiasm.

As explained, in Chapter 1 of [8], Marussi made a sharp distinction between *absolute* and *relative* entities. The former makes no use of coordinates, whereas the latter characterizes entities *totally* in terms of coordinates and invariance properties under change of coordinate systems. We will refer to the viewpoints as being *absolutist* and *relativist* in our discussion.

Having chosen the absolutist approach for vectors, Marussi naturally sought an analogous approach for tensors. The classical approach of G. Ricci-Curbastro (1853-1925) and T. Levi-Civita (1873-1941) to tensors was purely relativist, and in effect a complete absolutist theory of tensors did not exist. Marussi's solution was to adopt a provisional absolutist approach which was derived under the influence of the great French geometer E. Cartan (1869-1951) and presented in a little known monograph *Leçons sur le calcul vectoriel* (Blanchard, Paris 1930) of T. Ramos. Cartan had outlined the rudiments of such a viewpoint in the beginning chapters of his *Leçons sur la géométrie des espaces de Riemann* (Gauthier-Villars, Paris 1928) as a preliminary to his theory of exterior differential forms; however, Marussi made no use of the theory of differential forms. Unfortunately, even Cartan's approach was not well understood for many years and, without it being well-known, Marussi's tensorial methods appeared to many geodesists to be casual at best, and imprecise at worst.

It is ironic that, although, as we will see in §4, Marussi's choice of mathematical methods were based on sound and logical grounds, his vectorial methods were not understood because they were 'Italian', and a similar fate was suffered by his tensorial methods since they were 'non-Italian.' The net

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result was that Marussi's seminal ideas on intrinsic geodesy were formulated in a manner which failed to make them immediately comprehensible to his colleagues. Likewise, the matter was not helped by the fact that his most extensive presentation of his theory, i.e. [7], was written in Italian and published in an occasional series of memoirs which enjoyed a limited circulation.

5. Marussi's intrinsic geodesy from the viewpoint of contemporary differential geodesy.

In the previous section we have seen that Marussi's mathematical methods were unusual and did not lend themselves to an immediate acceptance of his theory. We now ask why he stubbornly held to them even long after it was clear to him that people found them difficult. It would be tempting to conclude that it was simply a matter of habit, nationalism, or perversity on his part, but upon examination none of these ring true. Moreover, he seemed content to follow his own way and as his introductory remarks in [1] reveal, he was generous in his praise of his younger colleagues, only one of whom ever used the homographic calculus.

The real -- and we think the convincing -- answer was that Marussi believed that the absolutist viewpoint was correct both aesthetically and conceptually even though it provided him only with a provisional theory of tensors. Although he freely employed coordinates in his work, indeed intrinsic geodesy is predicated on the existence of intrinsic coordinates, he felt that the basic concepts should be formulated *without* the use of coordinates. This is, of course, the classical geometric view in which synthetic geometry precedes analytic geometry and it has the weight of centuries to support it. However, in tensor calculus it is quite a modern idea which has won acceptance only over the last thirty years. The situation is well illustrated by the familiar question: What is a tensor? The

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relativist responds that it is an entity whose components transform in a particular way under a change of coordinates. However, the absolutist replies that such an answer merely describes a *property* possessed by the components of a tensor and it says nothing *per se* about what a tensor really is. The absolutist answer is that a tensor is an element in the tensor product of vector spaces, or, more generally, the tensor product of free modules of finite rank over an arbitrary commutative ring. This abstract answer is of a fairly recent vintage, and a full exposition of it was given only in 1948 by the Bourbakis in their *Éléments de mathématique*, Livre II, *Algebre* Chapitre 3. It took twenty years for it to become common knowledge in the mathematical community, and, in more recent times, by people working in mathematical and theoretical physics.

Hence, we maintain that Marussi used and thought in terms of the homographic calculus and his provisional tensor theory simply because they were the *only* absolutist formalisms available to him! In essence, he anticipated -- without knowing the details -- the modern so-called 'coordinate-free' approach to differential geometry and tensors. Thus, in retrospect, his intuitive and aesthetic feeling of how the mathematics should be done is vindicated although it is unlikely that he ever comprehended the level of abstraction on which the final answer would be given. Our contention that he employed mathematical methods which he regarded as provisional but on the right track also explains why his work shows virtually no influence of the relativist tensorial methods employed by Hotine. We believe that after outlining his basic approach, he did little to change his methodology. Indeed, his Erice lectures of 1974, [9], differ only slightly in their mathematical content from his earlier work [6]-[8], and in his last major paper, [10], he reverted to the homographic calculus as contained in [7]. Marussi had a deep and intuitive understanding of mathematical ideas which

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were strictly speaking beyond his grasp. But this is the true mark of genius in a physical scientist, and one shared with Newton, Maxwell, Einstein, and Dirac.

The question now is: How does Marussi's bold and beautiful conception of intrinsic geodesy stand today? In our view only one crucial question remains unanswered, and on its answer turns the entire mathematical structure of the theory. This question is simply whether an adequate supply of intrinsic coordinates actually exists. We will call the assertion of their existence the *Marussi Hypothesis*, and for purposes of discussion it is useful to delineate two forms of this hypothesis:

> *The Strong Form: All geodetic problems can be posed in terms of intrinsic coordinates;*
>
> *The Weak Form: Some geodetic problems can be posed in terms of intrinsic coordinates.*

The Strong Form states that coordinates are the appropriate manner of formulating mathematical geodesy, while the Weak Form asserts that coordinates may not be available. An alternate version of the Weak Form addresses the problem of determining under what circumstances coordinates do exist. The truth of the Strong Form would imply that the relativist form of tensor calculus is the appropriate mathematical formalism, whereas the truth of the Weak Form would lead us to differential forms or the leg calculus of Grafarend.

Only partial results relative to the general validity of the Marussi Hypothesis are known. First, we note that although it must have been of concern to him, Marussi never discussed it in his work. Moreover, for the most part the only intrinsic coordinate system which he considered was the local astronomical system (φ, λ, W) in which φ and λ are the astronomical latitude and longitude respectively, and W is the geopotential

function.

The discovery of non-holonomic coordinate/reference systems by E. Grafarend (1971) and N. Grossman (1974) show that the Strong Form of the Marussi Hypothesis is untenable. Grafarend discussed his discovery with Marussi and was amazed at how quickly he understood and accepted this rather unexpected classical result. Later in [10], Marussi himself employed a non-holonomic reference system. The situation is thus reduced to considering the Weak Form of the Marussi Hypothesis and essentially consists in inquiring whether in general one should expect to have intrinsic coordinates available. In other words, do intrinsic coordinates usually exist, or are they scarce in the sense that they occur only in very special situations? This question is non-trivial in that it requires an amalgamation of both mathematical and physical ideas. Mathematically the coordinates must exist not only at a point, but on a domain (i.e. an open connected subset) which is large enough to be useful, while physically they must be susceptible of measurement. Moreover, requirements must be consistent with each other in that neither, *a priori*, excludes the possibility of the other being satisfied. Borrowing the terminology introduced by J. Hadamard in his profound study of the Cauchy Problem, we may say that the Marussi Hypothesis requires that our coordinate/leg systems be *bien posé* both mathematically and physically. Since one can imagine different physical situations occurring in a single coordinate/leg system, only the mathematical part of the requirement is unambiguous: if the coordinate/leg system does not exist mathematically, then there is no possibility of doing any geodesy in it.

If it ultimately turns out that only the Weak Form of the Marussi Hypothesis is viable -- and there are strong hints that this will be the case -- then intrinsic geodesy must be subsumed into differential geodesy, and our classical preoccupation of thinking in terms of coordinates must be abandoned

as a chimera. We must then seek a mathematical formalism which efficiently deals with non-holonomic reference systems. This by no means diminishes the value of Marussi's vision of a new geodesy, anymore than Lagrangian or Hamiltonian dynamics diminishes Newtonian dynamics. It merely translates it into a new and challenging form.

In conclusion, Marussi has left to us as a legacy an exciting formulation of geodesy. He sketched the broad and bold outlines of a theory which demands that we re-think the tenets of classical geodesy in terms of new mathematical tools. Indeed, as Sir Alan Cook wrote in [1],

"Antonio Marussi left us thinking about the gravity field of the Earth and of geodesy in ways very different from those he found."

Just as his work challenged traditional geodetic thinking, his achievements challenge us to complete his vision of a new geodesy. In the concluding words of his inaugural lecture [11] at the first Symposium on Three-Dimensional Geodesy (Venezia, 1959) he said

"The cycle closes on itself and is renewed and reveals to our eyes new and distant horizons, which it will be our task to explore."

We can best honor his memory by completing, perfecting, and refining his ideas.

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investigation would have been impossible. Nevertheless, the conclusions and opinions expressed in this paper are my own and these individuals bear no responsibility for them. A more extensive analysis of the mathematical foundations of the Marussi-Hotine approach to geodesy will appear elsewhere [12].

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